1. WHAT IS ANALYSIS?

Objectives

(1) Deeper understanding of basic properties of sequences: sandwich theorem, nested interval theorem.

(2) Axiom of completeness is equivalent to the statement "every bounded non–increasing sequence converges".

(3) An overview of the problems arising in trying to do "calculus".

(4) Precise definition of continuity, and how it accords with intuitive ideas.

In 1A11 we studied *sequences* and *series* defined over the *real numbers*. The key technical idea was the following theorem: a bounded increasing sequence converges. This theorem is not true for the rational numbers and is the special feature of the reals that allows analysis to work. For the next part of analysis we need to have a different formulation of this theorem: an equivalent statement known as the Axiom of Completeness.

Axiom of Completeness. *The real numbers $\mathbb{R}$ are an ordered field (like $\mathbb{Q}$) satisfying one additional axiom:*

if a set S is bounded above, then it has a least upper bound. (**AOC**)

Notation. The least upper bound λ of the set S is written l.u.b.(S) or s.u.p.(S). It is uniquely characterized by the following properties:

(1) λ is an upper bound for S, so $s \in S \implies s \leq \lambda$.

(2) λ is the least number with (1): if $s \in S \implies s \leq \mu$ for some number μ, then $\mu \geq \lambda$.

An immediate consequence of this characterization is that the least upper bound of a set, if it exists, is unique (**Prove this**).

Example 1.1. Let $S = [0, 1]$. Then l.u.b.$(S) = 1$. This was proved in the lectures.

Example 1.2. Let $S = (0, 1)$. Then l.u.b.$(S) = 1$. This was proved in the lectures. The argument went as follows: firstly, $x \in S \implies x \leq 1$ so (1) holds. If (2) does not hold, then there is a number $\mu < 1$ that is an upper bound for S. But if $\mu < 1$, then the number $\frac{1}{2}(1 + \mu)$ is in S (**Check this**) but is larger than μ (**Check this too**), contradicting the assumption that μ is an upper bound for S. It follows that (2) holds.

Before showing that **AOC** is equivalent to the convergence of non–decreasing bounded above sequences, we give a useful new form of that result, the Nested Intervals Theorem.

Exercise. *If $a_n \leq \alpha$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, then $a \leq \alpha$. Show by example that even if $a_n < \alpha$ for all n, the $\leq$ cannot be replaced by $<$.*

Repeat for $\geq$.

Exercise. *Prove the Sandwich Theorem: if* $a_n \le x_n \le b_n$ *and* $\lim_{n\to\infty} a_n = \alpha = \lim_{n\to\infty} b_n$ *then* $\lim_{n\to\infty} x_n = \alpha$.

Theorem 1.3. *Let* $[a_n, b_n]$ *be a closed interval for each* $n \in \mathbb{N}$. *Assume that*

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots \supset [a_n, b_n] \supset \dots$$

(the intervals are "nested").
(a) Then there is a number $c \in [a_n, b_n]$ *for all* $n \in \mathbb{N}$.
(b) Moreover, if we also have

$$\lim_{n\to\infty} (b_n - a_n) = 0$$

then there is only one number c *lying in all the intervals* $[a_n, b_n]$.
(c) Finally, if $x_n \in [a_n, b_n]$ *for all* n *and* $\lim_{n\to\infty}(b_n - a_n) = 0$ *then*

$$\lim_{n\to\infty} x_n = c.$$

Proof. (a) First, notice that the intervals being nested means exactly that the sequence (a_n) is non–decreasing and the sequence (b_n) is non–increasing. Since both sequences are contained in the set $[a_1, b_1]$ they are bounded, so must converge. Let

$$a = \lim_{n\to\infty} a_n; \quad b = \lim_{n\to\infty} b_n.$$

Now $a_n \le a$ for all n and $b \le b_n$ for all n. It follows (since the intervals are nested) that $a_p \le b_q$ for all p and q. Hence (use the exercise above) $a \le b$. If $c \in [a, b]$ then $c \in [a_n, b_n]$ for every n.
(b) If $c \in [a_n, b_n]$ for every n then $a_n \le c \le b_n$ for every n. Since $\lim_{n\to\infty}(b_n - a_n) = 0$, we have

$$\begin{aligned} a = \lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} (b_n - a_n) \\ &= \lim_{n\to\infty} (a_n + (b_n - a_n)) = \lim_{n\to\infty} b_n = b. \end{aligned}$$

It follows that $a = b = c$.
(c) We have $a_n \le x_n \le b_n$ for all n so by the sandwich theorem x_n converges to $a = b = c$. □

Theorem 1.4. *The Axiom of Completeness is equivalent to the statement "every bounded increasing sequence converges".*

Proof. First assume the **AOC** and let (x_n) be a bounded increasing sequence. Thus there is a number S with $x_n \le S$ for all n. Using **AOC**, we know that the *set* $A = \{x_n\}$ must have a least upper bound $\lambda = \sup(A)$ say. We claim that $\lim_{n\to\infty} a_n = \lambda$. To see this, notice that for any $\epsilon > 0$, there must be an element of the set A, a_N say, with the property that $a_N > \lambda - \epsilon$. (If not, λ would not be the least upper bound). Then by the non–decreasing property, $n > N \implies a_n \ge a_N > \lambda - \epsilon$, so $n > N \implies |a_n - \lambda| < \epsilon$.

Conversely, assume that every non–decreasing bounded above sequence converges and let A be a set that is bounded above by S say. Choose numbers a_1 and b_1 as follows: a_1 lies in A, and b_1 is any upper bound for A. Notice that

$$[a_1, b_1] \cap A \neq \emptyset; \quad x \leq b_1 \text{ for all } x \in A.$$

Now chop the interval $a_1, b_1]$ in half: consider the two intervals

$$[a_1, \frac{a_1 + b_1}{2}], \text{ and } [\frac{a_1 + b_1}{2}, b_1].$$

If the second interval contains some points of A, then let $a_2 = \frac{a_1+b_1}{2}$, $b_2 = b_1$. If not, let $a_2 = a_1$ and $b_2 = \frac{a_1+b_1}{2}$. By construction b_2 is still an upper bound for A, so

$$[a_2, b_2] \cap A \neq \emptyset; \quad x \leq b_2 \text{ for all } x \in A.$$

Continue this bisection process to construct a sequence of nested intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset \dots$$

with the property that $lim_{n \to \infty}(b_n - a_n) = 0$. In addition, for every $n \in \mathbb{N}$,

$$[a_n, b_n] \cap A \neq \emptyset; \quad x \leq b_n \text{ for all } x \in A.$$

By the nested interval theorem, there is a unique real number λ with the property that

$$\lambda = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n.$$

Claim that $\lambda = \sup(A)$.

To see this, let $x \in A$. Then by construction, $x \leq b_n$ for all n so $x \leq \lambda = \lim_{n \to \infty} b_n$. It follows that λ is an upper bound for A.

On the other hand, if μ is any upper bound for A, then since $[a_n, b_n] \cap A \neq \emptyset$ for all n, it follows that $a_n \leq \mu$ for all n so $\lambda = \lim_{n \to \infty} a_n \leq \mu$, proving that λ is indeed the least upper bound. $\square$

A simple remark that will be useful later is the following.

Lemma 1.5. *If $a < b$ are real numbers, then there is a rational number r and an irrational number s with $a < r < b$, $a < s < b$.*

Notice that by applying Lemma 1.1 repeatedly we may find between any two distinct real numbers infinitely many rationals and infinitely many irrationals.

In this course we shall use ideas from 1A11 to study *functions*. We have already seen at the beginning of 1A11 a discussion about the function $f(x) = x^2 - 2$; this function has the property that $f(0) < 0$ and $f(2) > 0$. Can we conclude that there is a number ξ with $0 < \xi < 2$ and $f(\xi) = 0$? In Section 4 we shall single out a class of functions for which this kind of argument works – the *continuous* functions. We know already that there is no such ξ in the rationals, so the notion of continuity must involve properties that distinguish the real numbers from the rationals: the Axiom of Completeness, or equivalently results on *limits* of sequences and series.

After we have developed the basic machinery of *limits of functions*, *continuous functions*, and some more material on *sequences and subsequences*, we prove the Intermediate Value Theorem (which, for example, asserts the existence of ξ).

The second half of the course is devoted to a rigorous development of *calculus*: what does it mean to say a function is *differentiable* or *integrable*? What general properties hold for such functions?

AREAS

At school you will have encountered the statement that definite integrals give the *area* under a curve. We don't really have any definition of area, so let's see how far we can get from a reasonable starting point:

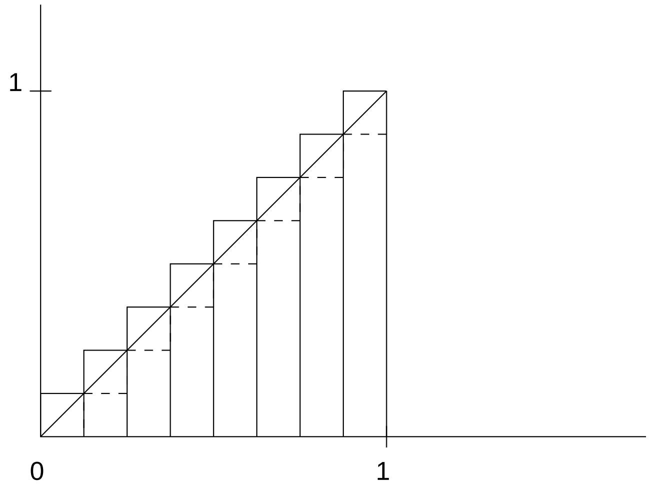
(1) the area of a rectangle with width a and height b is $a \times b$,

(2) the area of the union of two non–overlapping rectangles is the sum of their areas, and

(3) if A is a subset of B then the area of A is less than or equal to the area of B.

Using this, let us try to find the area under the curve $y = x$ between $x = 0$ and $x = 1$; call this set A.

Fix a natural number n. Split the interval $[0, 1]$ into n equal sub–intervals by the *partition* $\mathcal{P} = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} = 1\}$. Associated with the partition $\mathcal{P}$ is a staircase–shaped set $U(\mathcal{P})$ (solid lines) which *contains* A, and another staircase–shaped set $L(\mathcal{P})$ (dotted lines) which is *contained in* A. What is the area of $U(\mathcal{P})$?



First rectangle is $\frac{1}{n} \times \frac{1}{n}$,
Second rectangle is $\frac{1}{n} \times \frac{2}{n}$,
and so on until
nth rectangle is $\frac{1}{n} \times \frac{n}{n}$.

Thus the total area of the set $U(\mathcal{P})$ is:

$$\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} = \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

A similar calculation shows that the total area of $L(\mathcal{P})$ is:

$$0 + \frac{1}{n^2} + \cdots + \frac{n-1}{n^2} = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

Applying (3) we conclude that

$$\frac{1}{2}\left(1 - \frac{1}{n}\right) \leq \text{Area of } A \leq \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

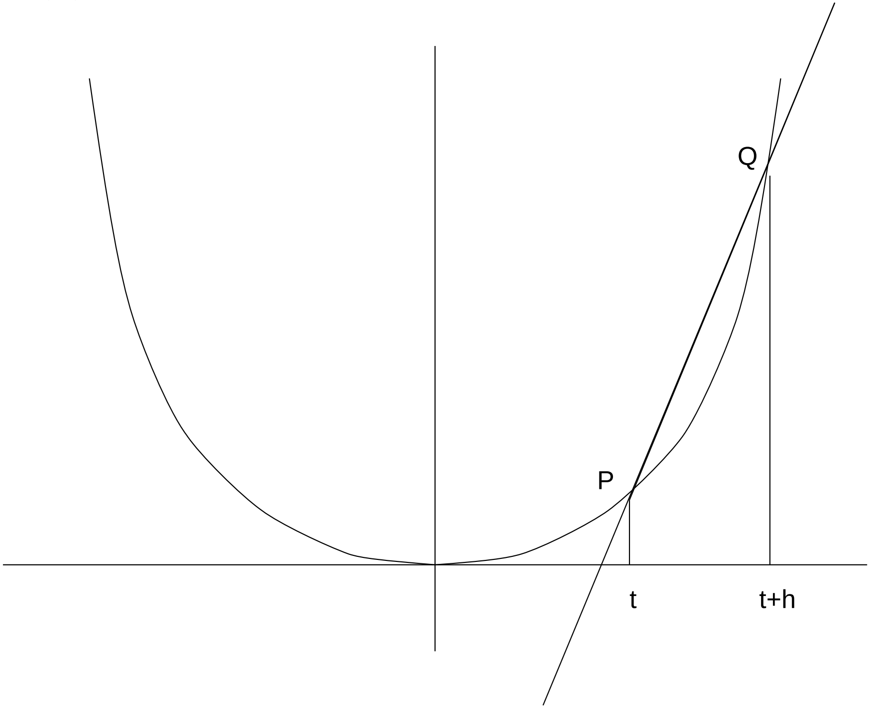
As n goes to infinity it is therefore reasonable to conclude that the area of A is $\frac{1}{2}$. However, there are several problems left open. Does the answer depend on the choice of the partition P? Does this method work for any function? On the first homework sheet you will be asked to attempt such a calculation for the function g defined by $g(x) = 0$ if x is irrational, and $g(x) = 1$ if x is rational.

TANGENTS

A *tangent* to a curve $y = f(x)$ at the point $(t, f(t)$ is a straight line that touches the curve at that point and has the *same slope* as the curve – whatever that means.

Consider the curve $y = x^2$: we shall find the tangent to the curve at the point (t, t^2) by a limiting process.

Call the point (t, t^2) P, and let Q be another point on the curve, say $Q = (t + h, (t + h)^2)$.



The slope of the line through P and Q is given by

$$\frac{f(t+h) - f(t)}{(t+h) - t} = \frac{(t+h)^2 - t^2}{h} = 2t + h,$$

so as $h \to 0$, i.e. as Q moves to P, we expect in the limit to find that the slope at (t, t^2) is exactly $2t$.

Again, questions remain. Does this process work for any curve? That is, is there a unique tangent at every point? See homework sheet 1 for more details on the following example. Let $g(x) = |x|$, and try to use the above method to find the slope of the tangent at $x = 0$. You will find that $\frac{g(h)-g(0)}{h}$ is $+1$ when $h > 0$ and -1 when $h < 0$, so it does not converge to any fixed number as $h \to 0$.

If the above process does work, then we shall say that the function f is *differentiable* and the function $f'(x) =$ slope of that tangent at $(x, f(x))$ is called the *derivative* of f.

FUNDEMANTAL THEOREM

We shall see later that, under certain conditions, the process of finding areas under curves (*integration*) and the process of finding slopes of tangents (*differentiation*) are inverse processes. This was first pointed out by Newton, and various precise statements of this result are called the Fundamental Theorem of Calculus.

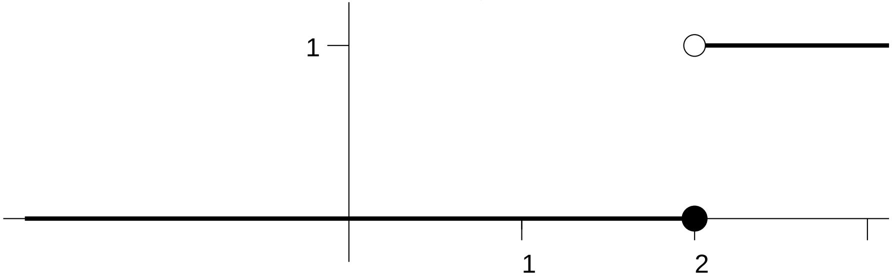
CONTINUITY

The function $g(x) = |x|$ has no derivative at $x = 0$, but it does have some good properties: there are no jumps or breaks in the graph of g for example. How may we express precisely this property (which is roughly expressed by saying that the graph of g may be drawn without lifting the pen from the paper)? We begin by formulating exactly the property that the graph of a function f *has no jumps*.

The function f has a *jump* at $x = a$ if the graph of f breaks at $x = a$ and jumps to a different level. Here is the graph of the function

$$f(x) = \begin{cases} 0 & x \leq 2 \\ 1 & x > 2. \end{cases}$$

Notice that the function has a jump of height 1 at the point $x = 2$.



The jump may be detected by noticing that there are values of x *arbitrarily close to, but not equal to, 2* for which $|f(x) - f(2)| \geq 0.9$ say. Of course there is nothing special about the height of the jump or the location of the jump, so we make the following definition:

Definition. The function f has a jump at the point $x = a$ if there is an $\epsilon > 0$ such that for every $\delta > 0$, there exists an $x \neq a$ with $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$.

Writing this out in quantifiers: f has a jump at the point $x = a$ if

$$\exists\ \epsilon > 0 \text{ such that } \forall \delta > 0\ \exists\ x, |x - a| < \delta \text{ for which } |f(x) - f(a)| \geq \epsilon.$$

Pushing this one stage further, the function f is said to be *discontinuous* if it has a jump somehere. So f is discontinuous if

$$\exists\ a,\ \exists\ \epsilon > 0 \text{ such that } \forall\ \delta > 0\ \exists\ x, |x - a| < \delta \text{ for which } |f(x) - f(a)| \geq \epsilon.$$

The negation of being discontinuous will be called continuity:

Definition 1.2. The function f is *continuous* if

$$\forall\ a,\ \forall\ \epsilon > 0\ \exists\ \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

In sheet 1 you will check that the function $f(x) = |x|$ does have this property despite being non–differentiable. The definition of continuity will appear again in several different forms, and we shall see that the continuous functions are exactly those that satisfy the Intermediate Value Theorem.

Things you should have learned in this section

(1) How to prove and use the sandwich and nested intervals theorems.
(2) How to manipulate the supremum and infimum of a set.
(3) The definition of continuity for a function at a point.

TBW/1A22

2. The Limit of a Function at a Point

Objectives

(1) To develop a precise language for talking abour functions: domain, image, bounded.

(2) To define the limit of a function at a point using punctured neighbourhoods.

(3) To relate limits of functions to limits of sequences.

Recall the following standard notation for functions: a *function* is a triple $f : X \to Y$. The set X is the *domain* of f, and Y the *codomain*. For a subset $A \subset X$, the *image of A under f* is the set $f(A) \subset Y$. For a set $B \subset Y$, the *pre–image of B under f* is the set $f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subset A$.

Exercise. If B is non–empty, there is no reason to suppose that $f^{-1}(B)$ is non–empty. Why not?

If the function f is a *bijection* then it has an inverse function $f^{-1} : Y \to X$.

In this course all the function we deal with will be *real–valued functions of a real variable*, which means the sets X and Y will be subsets of $\mathbb{R}$.

Functions may be added, multiplied, divided (carefully!) and multiplied by real numbers in the obvious way:

$$(f+g)(x) = f(x) + g(x),$$
$$(fg)(x) = f(x) \times g(x),$$
$$(f/g)(x) = f(x)/g(x)$$

(so long as $g(x) \neq 0$), and

$$(\lambda f)(x) = \lambda \times f(x),$$

where $\lambda \in \mathbb{R}$.

Exercise. Given functions $f : X \to Y$, and $g : Z \to W$, what is the domain of $f + g$? What about fg and f/g?

Definition 2.1. A function $f : D \to \mathbb{R}$ is said to be

(1) *bounded above on* $S \subset D$ if $\exists\ R$ such that $f(x) \leq R$ for $x \in S$,
(2) *bounded below on* $S \subset D$ if $\exists\ R$ such that $f(x) \geq R$ for $x \in S$, and
(3) *bounded* on $S \subset D$ if $\exists\ K$ such that $|f(x)| \leq K$ for $x \in K$.

Definition 2.2. A *neighbourhood* of a is an open interval of the form

$$(a - \delta, a + \delta),$$

where $\delta > 0$. A *punctured nieghbourhood* of a is a set of the form

$$(a - \delta, a) \cup (a, a + \delta)$$

for some $\delta > 0$.

Notice that a point is ALWAYS contained in any neighbourhood of itself, and is NEVER contained in any punctured neighbourhood of itself.

Example 2.3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x+1}{x-1} & x \neq 1 \\ 0 & x = 1. \end{cases}$$

Draw the graph of f; from the graph we expect the following to be true.

1. f is bounded on $[2, 5]$,
2. f is bounded above on $(-\infty, 1)$,
3. f is bounded below on $(1, \infty)$,
4. f is not bounded on any neighbourhood of 1.

Proof of (1): For $x \in [2, 5]$, it is clear that $3 \leq x + 1 \leq 6$ and $1 \leq x - 1 \leq 4$, so $|\frac{x+1}{x-1}| \leq \frac{6}{1} = 6$.
Proof of (2): For $x < 1$, $x - 1 < 0$, so $x + 1 > x - 1$ implies that $\frac{x+1}{x-1} < 1$. It follows that f is bounded above by 1 on the set $(-\infty, 1)$.
Proof of (3): For $x > 1$, $x - 1 > 0$, so $x + 1 > x - 1$ implies that $\frac{x+1}{x-1} > 1$. It follows that f is bounded below by 1 on the set $(1, \infty)$.
Proof of (4): If (4) is not true, then for some neighbourhood of 1, f is bounded: that is, there is a $\delta > 0$ and a number $K > 0$ with the property that

$$x \in (1 - \delta, 1 + \delta) \implies |f(x)| < K.$$

We want to show that this assumption leads to a contradiction, by exhibiting an x with the following properties. First, $1 < x < 1 + \delta$, and second, $|f(x)| > K$. A calculation shows that $1 < x < \frac{1+2K}{2K-1}$ implies that $f(x) > 2K$, so we can complete the argument by choosing $x = \min\{1 + \delta, \frac{1+2K}{2K-1}\}$.

Definition 2.4. Let D be a set containing some punctured neighbourhood of a point a, and let $f : D \to \mathbb{R}$ be a function. Then the *limit of $f(x)$ as x tends to a is ℓ* if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \implies \quad |f(x) - \ell| < \epsilon.$$

This is normally written as

$$f(x) \to \ell \quad \text{as} \quad x \to a,$$

or

$$\lim_{x \to a} f(x) = \ell.$$

Notice that f need not be defined at the point $x = a$. Even if it is defined there, the value of $f(a)$ need not be equal to $\lim_{x \to a} f(x)$.

Example 2.5. Let

$$f(x) = \begin{cases} \frac{x}{1+x} & x \neq 1 \\ 0 & x = 1. \end{cases}$$

Prove that $\lim_{x \to 1} f(x)$ exists and is equal to $\frac{1}{2}$.

To prove this, we must first understand how to estimate the size of the expression

$$\left|\frac{x}{1+x}-\frac{1}{2}\right|,$$

ideally in terms of the quantity $|x-1|$. Now

$$\frac{x}{1+x}-\frac{1}{2}=\frac{2x-(1+x)}{2(1+x)}=\frac{1-x}{2(1+x)}.$$

If $|x-1|<\frac{1}{2}$, then $1+x>\frac{3}{2}$, so by the above calculation $|\frac{x}{1+x}-\frac{1}{2}|<\frac{1}{3}|x-1|$.

From that calculation, it is now easy to write down the required statement. Given any $\epsilon>0$, choose $\delta<\min\{\frac{1}{2},3\epsilon\}$. Then $0<|x-1|<\delta \implies |f(x)-\frac{1}{2}|<\epsilon$.

Example 2.6. Let

$$g(x)=\begin{cases} x & x>1,\\ 0 & x\le 1.\end{cases}$$

Then the limit of $g(x)$ as x approaches 1 does not exist.

We will have an easy way to prove this soon, but for now let us argue directly from Definition 2.4. If the limit exists, then for some ℓ we have:

$$\forall\ \epsilon>0\ \exists\delta>0 \text{ such that } 0<|x-1|<\delta \implies |g(x)-\ell|<\epsilon.$$

Assume that the limit does exist, so that this statement is true. Choose $\epsilon=\frac{1}{4}$, and notice that

$$|g(x)-\ell|<\tfrac{1}{4},\ |g(y)-\ell|<\tfrac{1}{4} \implies |g(x)-g(y)|<\tfrac{1}{2}.$$

(Make sure you see why this is so). Now, under the assumption that the limit exists, and choosing the δ corresponding to $\epsilon=\frac{1}{2}$, we have a $\delta>0$ with the property that

$$0<|x-1|<\delta \implies |g(x)-\ell|<\tfrac{1}{2}.$$

Let $x=1+\frac{\delta}{2}$, and $y=1-\frac{\delta}{2}$. Putting all of the above together, we conclude that:

$$|g(x)-g(y)|<\tfrac{1}{2},$$

which is clearly impossible since $g(x)=x>1$ and $g(y)=0$.

Work through Example 2.6 carefully, and make sure you understand that it is just a complete rendition of a simple idea that we have already encountered with sequences. If $g(x)$ is approaching a limit as x approaches 1, then for x and y very close to 1, $g(x)$ and $g(y)$ must both be very close to whatever the limit is, and therefore very close to each other. On the other hand, it is clear from the definition of the function g that there are arbitrarily close–together pairs x, y, close to 1, with the property that $g(x)$ is not close to $g(y)$.

Theorem 2.7. *If a function has a limit at a point, then that limit is unique.*

Proof. (First ask yourself how would prove that a convergent sequence can have only one limit.)

Let f be a function defined on some punctured neighbourhood of a point c, and assume that $f(x) \to \ell$ as $x \to c$.

Now assume in addition that $f(x) \to \ell'$ as $x \to c$: our aim is to deduce that ℓ must be equal to ℓ'.

If $\ell \neq \ell'$, then there is a positive number ϵ with the property that $2\epsilon < |\ell - \ell'|$. On the other hand, by assumption we may find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_1 \implies |f(x) - \ell| < \epsilon,$$

and

$$0 < |x - c| < \delta_2 \implies |f(x) - \ell'| < \epsilon.$$

Choose x so close to c that $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$, so

$$|f(x) - \ell| < \epsilon \text{ and } |f(x) - \ell'| < \epsilon.$$

By the triangle inequality, this implies that

$$|\ell - \ell'| < 2\epsilon,$$

which contradicts our choice of ϵ.

The only possible conclusion is that $\ell = \ell'$. ☐

Again, work through the details and make sure that you understand the ideas behind the proof.

So far all the proofs about limit of functions seem to closely resemble analogous results for sequences. The next theorem shows how the notions are related: the idea is again simple, but the details are a little tricky.

First, some notation. Let $c \neq x_n \to c$ mean the sequence $(x_n)_{n\in\mathbb{N}}$ converges to c and for each n, $x_n \neq c$.

Now consider the following properties of the function f defined on some punctured neighbourhood D of the point c:
[1] $\lim_{x\to c} f(x) = \ell$,
[2] if $c \neq x_n \to c$ in D, then $f(x_n) \to \ell$.

Notice that [2] is expressed entirely in terms of sequences (in particular, if (x_n) is a sequence then so is $(f(x_n))$).

Theorem 2.8. [1] *is equivalent to* [2].

Proof that [1] $\implies$ [2]**.** As we know, [1] means for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \quad \implies \quad |f(x) - \ell| < \epsilon.$$

Let (x_n) be a sequence with $c \neq x_n \to c$, and let $\epsilon > 0$ be given. Then choose $\delta > 0$ according to [1], and let N be chosen to have the property that

$$n \geq N \implies 0 < |x_n - c| < \delta;$$

then we have found N such that

$$n \geq N \implies |f(x_n) - \ell| < \epsilon,$$

showing that $f(x_n) \to \ell$. ☐

Proof that [2] $\implies$ [1]**.** To do this, we shall prove the contrapositive statement, "not[1]" implies "not[2]".

Assume therefore that

$$\exists\ \epsilon > 0 \text{ s.t. } \forall\ \delta > 0\ \exists\ x, 0 < |x - c| < \delta \text{ such that } |f(x) - \ell| \geq \epsilon.$$

Choose in turn $\delta = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ to find a sequence $x_1, x_2, x_3, \ldots$ with the property that

$$0 < |x_n - c| < \frac{1}{n} \quad \textit{and} \quad |f(x_n) - \ell| \geq \epsilon.$$

The first part of this shows that $c \neq x_n \to c$, and the second shows that $f(x_n)$ is not converging to ℓ, so [2] is not true. ☐

Using Theorem 2.8, most of the basic properties of limits of functions follow easily from the corresponding results for sequences. There are some exceptions though – Theorem 2.9 is most easily proved using Definition 2.4.

Theorem 2.9. *If a function has a limit at the point c, then the function is bounded on some punctured neighbourhood of c.*

Proof. Suppose that $f(x) \to \ell$ as $x \to c$. Use $\epsilon = 1$ in Definition 2.4, and choose δ so that $0 < |x - c| < \delta \implies |f(x) - \ell| < 1$. Then, for all x with $0 < |x - c| < \delta$, $|f(x)| = |(f(x) - \ell) + \ell < 1 + |\ell|$, showing that f is bounded in a punctured neighbourhood of c. ☐

Theorem 2.10. *Assume that $f(x) \to \ell$ and $g(x) \to m$ as $x \to c$. Then*

1. *$f(x) + g(x) \to \ell + m$ as $x \to c$,*
2. *$f(x)g(x) \to \ell m$ as $x \to c$,*
3. *for $\lambda \in \mathbb{R}$, $\lambda f(x) \to \lambda \times \ell$ as $x \to c$, and*
4. *if $m \neq 0$ then $f(x)/g(x) \to \ell/m$ as $x \to c$.*
5. *if $f(x) \leq g(x)$ then $\ell \leq m$.*

Proof. These follow from the corresponding statements about sequences. As an exercise, write out proofs without using sequences – as with sequences, you will need Theorem 2.9. ☐

Example 2.11. Show that $\lim_{x\to 2} \frac{x^3 - 2x^2 + 8}{x^2 + 7x + 3}$ exists and find the limit.

We have shown that, for example, $\lim_{x\to 2} x = 2$. It follows by Theorem 2.10 that $\lim_{x\to 2} x^3 = 8$, $\lim_{x\to 2} x^3 - 2x^2 = 8 - 2\lim_{x\to 2} x^2 = 0$ and so on, showing that the limit exists and equals $\frac{8}{21}$.

Example 2.12. In practice it is more important to be able to find limits like $\lim_{x\to 2} \frac{x^3 - 2x^2 + 3x - 6}{x^2 + 7x - 18}$, or even worse $\lim_{x\to 2} \frac{x^3 - 2x^2 + 3x - 6}{\tan(x - 2)}$. Make sure you understand why neither of these is trivial, and see if you can work out what the limits are (use your calculator if need be).

Example 2.13. Prove the Sandwich Theorem: if $h(x) \leq f(x) \leq g(x)$ and $h(x) \to \ell$, $g(x) \to \ell$ as $x \to c$, then $f(x) \to \ell$ also.

Fill in the details: the key idea is that if $|h(x) - \ell| < \epsilon$, and $|g(x) - \ell| < \epsilon$, then $|h(x) - g(x)| < 2\epsilon$ and so $|f(x) - \ell| < 3\epsilon$.

Example 2.14. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 1 & x = 0. \end{cases}$$

Prove that $\lim_{x\to 0} f(x)$ does not exist.

To do this, use the sequential equivalent definition of the existence of a limit in Theorem 2.8. Let $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+\frac{1}{2})\pi}$. Then $0 \neq x_n \to 0$ and $0 \neq y_n \to 0$, but $f(x_n) = 0 \to 0$ while $f(y_n) = 1 \to 1$, showing that the limit does not exist.

Things you should have learned in this section

(1) The definition of domian, co-domain, image, boundedness for functions.

(2) The exact definition of what it means for a function to have a limit at a point.

(3) How to prove theorems about limits of functions using results about sequences.

3. OTHER LIMITS OF FUNCTIONS

Objectives

(1) To formulate various “improper” limits: one–sided, limits at infinity and so on.

It is sometimes useful to have definition for slightly different limits. Here we consider *one–sided limits* and *tending to infinity*.

Definition 3.1. If f is defined on the set $(c, c+\eta)$ for some $\eta > 0$, then ℓ is the *limit of f from the right*, written $f(x) \to \ell$ as $x \to c^+$ or $\lim_{x\to c^+} f(x) = \ell$ if, for any $\epsilon > 0$ there exists $\delta > 0$ such that $0 < x - c < \delta \implies |f(x) - \ell| < \epsilon$.

Of course there is a “from the left” version as well, written $\lim_{x\to c^-}$.

Theorem 3.2. *Let f be defined in a punctured neighbourhood of c. Then*

$$\lim_{x\to c} f(x) = \ell$$

if and only if $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = \ell$.

This Theorem (whose proof is trivial) is sometimes useful in showing that a limit does not exist: use it for Example 2.6 as an exercise.

Definition 3.3. *Let f be defined on some neighbourhood of c. Then $f(x)$ tends to infinity as x tends to c, or* $\lim_{x\to c} f(x) = \infty$ *if, for every $K \in \mathbb{R}$ there is a $\delta > 0$ such that* $0 < |x - c| < \delta \implies f(x) \geq K$.

Exercise 3.4. Prove that $\lim_{x\to c} |f(x)| = \infty$ if and only if $\lim_{x\to c} \frac{1}{f(x)} = 0$.

Of course there are other combinations: to make sure you understand this material, write out the definition of the statement $\lim_{x\to c^-} f(x) = -\infty$.

Example 3.5. Fill in the details of the following. Let $f(x)$ be defined by

$$f(x) = \frac{2x}{x-1} - \frac{3x}{x^3 - 8}$$

for $x \neq 1, 2$. Then $\lim_{x\to 1^+} f(x) = +\infty$, $\lim_{x\to 1^-} f(x) = -\infty$, $\lim_{x\to 2^+} f(x) = -\infty$, and $\lim_{x\to 2^-} f(x) = +\infty$.

Things you should have learned in this section

(1) How to prove statements involving one–sided limits and limits involving infinity.

(2) How to use these ideas to help sketch graphs of functions.

4. Continuity – Basic Results

Objectives

(1) To formulate an exact notion of continuity at a point for functions.
(2) To develop an "algebra" for continuous functions.

Recall Definition 1.2, which characterised those functions that have no jumps.

Definition 4.1. Suppose that $f : D \to \mathbb{R}$ is defined in some neighbourhood of the point c. Then f is *continuous at* c if, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Notice the key difference between this property and the statement "$f(x)$ has a limit as $x \to c$". For continuity, we include the point c, and deal with neighbourhoods rather than punctured neighbourhoods.

A function is said to be *continuous on a set* if it is continuous at each point of the set, and simply *continuous* if it is continuous at each point of the domain.

Theorem 4.2. *Suppose that $f : D \to \mathbb{R}$ is defined in some neighbourhood of the point c. Then f is continuous at c if and only if*

$$\lim_{x \to c} f(x) = f(c).$$

Remember that the statement means two things: firstly, that the limit of $f(x)$ as x approaches c *exists*, secondly that it *is equal to* $f(c)$.

Proof of Theorem 4.2. A careful comparison of Definition 4.2 and Definition 2.4 shows that continuity at the point c is exactly the same property as $\lim_{x \to c} f(x) = f(c)$. Check that you understand what happens to the special point c in this comparison. ☐

All the results of Section 2 may now be applied to deduce properties of continuous functions. For some, the proof is given; for the later ones you should provide the proof.

Theorem 4.3. *Suppose that $f : D \to \mathbb{R}$ is defined in some neighbourhood of the point c. Then f is continuous at c if and only if for every sequence $x_n \to c$ in D,*

$$\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n) = f(c).$$

Proof. This follows at once from Theorem 2.7 – again, make sure you see what happens to the special point c. ☐

Notice that Theorem 4.3 is an example of a special situation in which a *limit commutes*. On the homework you will see some situations involving discontinuous functions where a limit does not commute with the function.

Theorem 4.4. *If f is continuous at the point c, then f is bounded on some neighbourhood of c.*

Proof. By Theorem 2.8, f is bounded on some punctured neighbourhood of c. It is also defined at c, so we are done. $\square$

Theorem 4.5. *Suppose that f and g are continuous at c. Then so are*

(1) $f + g$,
(2) λf *for* $\lambda \in \mathbb{R}$,
(3) $f \times g$, *and*
(4) f/g *if* $g(c) \neq 0$.

Proof. Apply Theorem 2.8 $\square$

Examples 4.6. (1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $f(r) = 0$ whenever r is rational. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

This is best done using Theorem 4.3: fix a real number x, and let $r_1, r_2, r_3, \ldots$ be a sequence of rational numbers converging to x. Then, by assumption $f(r_n) = 0$ for all n, so by continuity

$$f(x) = f(\lim_{n\to\infty} r_n) = \lim_{n\to\infty} f(r_n) = 0.$$

(2) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $g(0) > 1$. Prove that $g(x) > 0.9$ for all x in some neighbourhood of 0.

For this problem, work directly from Definition 4.1. Take $\epsilon = 0.1$. Then by continuity, there is a $\delta > 0$ for which

$$|x - 0| < \delta \implies |g(x) - g(0)| = |g(x) - 1| < 0.1,$$

so that $g(x) > 0.9$ for all $x \in (-\delta, \delta)$.

(3) Theorem 4.5 shows that, for example functions like $f(x) = \frac{x^3-4x^2+7}{3x^2+4}$ are continuous wherever they are defined.

One last important "basic" result about continuous functions.

Theorem 4.7. *If $g : A \to B$ is continuous, and $f : B \to \mathbb{R}$ is continuous, then so is the composition $fg : A \to \mathbb{R}$ defined by $f \cdot g(x) = f(g(x))$.*

Proof. Let c be a point in A, and let $x_n \to c$ in A. Then by continuity of g, $g(x_n) \to g(c)$ in B, so by continuity of f, $f(g(x_n)) \to f(g(c))$. $\square$

Remark 4.8. If a function is defined on a set of the form $[c, c+\eta)$ say, then there is an obvious notion of the function being *continuous from the right at c* and so on. Write down the definition of this property, and the corresponding continuity from the left. We shall need this to make sense of statements like "f is continuous on the set $[a, b]$", which means

$$\forall\ c \in [a, b],\ \forall\ \epsilon > 0,\ \exists\ \delta > 0 \text{ such that}$$

$$x \in (c-\delta, c+\delta) \cap [a,b] \implies |f(x) - f(c)| < \epsilon.$$

Things you should have learned in this section

(1) The definition of the statement "f is continuous at a".
(2) The definition of the statement "f is continuous".
(3) How to prove theorems about continuous functions using known results on sequences.

TBW/1A22

5. CONTINUITY – INTERMEDIATE VALUE THEOREM

Objectives

(1) To prove that bounded sequences have convergent subsequences.

(2) To apply (1) to show that continuous functions on closed bounded intervals are bounded and attain their bounds.

(3) To formulate and prove the Intermediate Value Theorem.

We begin with the *Bolzano–Weierstrass Theorem.*

Theorem 5.1. *If (x_n) is a sequence of points with $x_n \in [a, b]$ for every n, then there is a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ that converges to some point $c \in [a, b]$.*

The word subsequence means that the indices $n_1, n_2, \ldots$ are strictly increasing: $n_1 < n_2 < \ldots$.

We have seen already in 1A11 a result that guaranteed convergence: a non–decreasing sequence that is bounded above must converge.

Proof of Theorem 5.1. Recall Theorem 2.10 from 1A11: if (a_n) is monotone non–decreasing or non–increasing and bounded, then it is convergent.

It is therefore sufficient to show the following.

Lemma. *Any sequence (a_n) contains a subsequence (a_{n_j}) which is either non–decreasing or non–increasing.*

Proof of Lemma. Let us say that n is a *peak point* of (x_n) if $a_m < a_n$ for all $m > n$. There are two possibilities:

CASE 1. *The sequence has infinitely many peak points.* In this case, let $n_1 < n_2 < n_3 < \ldots$ be the peak points, so that $a_{n_1} > a_{n_2} > a_{n_3} > \ldots$ is a non–increasing subsequence.

CASE 2. *The sequence has only finitely many peak points.* Let n_1 be chosen greater than any of the peak points. Since n_1 is *not* a peak point, there must be some $n_2 > n_1$ for which $a_{n_2} \geq a_{n_1}$. Since n_2 is not a peak point, there is some $n_3 > n_2$ for which $a_{n_3} \geq a_{n_2}$. Continuing, we arrive at the non–decreasing subsequence $a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \ldots$. □

The Bolzano–Weierstrass Theorem is now clear: from the sequence (x_n) extract a non–decreasing or non–increasing subsequence. Since the original sequence lies in $[a, b]$, the subsequence is bounded and hence convergent.

The only thing left to check is that the limit of the convergent subsequence is also in $[a, b]$, but this is clear: if $t \notin [a, b]$ then t is some positive distance from all elements of $[a, b]$, and hence from every term of the subsequence, and therefore cannot be the limit. □

Theorem 5.2. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f is bounded.*

Notice that two things are needed for the result: the function must be continuous, and the domain must be restricted. For example,

$$f(x) = \begin{cases} n & x \in [\frac{1}{n+1}, \frac{1}{n}) \text{ for } n \in \mathbb{N}, \\ 0 & \text{for } x = 0, 1, \end{cases}$$

is a discontinuous function on $[0,1]$ that is not bounded.

On the other hand, $g(x) = \frac{1}{x}$ is a continuous function on $(0,1]$ that is not bounded.

Proof of Theorem 5.2. The function f is bounded on $[a,b]$ means

$$\exists\ M,\ \forall\ x \in [a,b], |f(x)| \le M.$$

Assume (for the purposes of a contradiction) that f is continuous on $[a,b]$ but is not bounded:

$$\forall\ M,\ \exists\ x \in [a,b], |f(x)| > M.$$

It follows that there is a sequence of points $x_1, x_2, \dots$ in $[a,b]$ with the property that

$$f(x_n) \ge n$$

for each $n \in \mathbb{N}$. By the Bolzano–Weierstrass Theorem, the sequence (x_n) has a subsequence $x_{n_1}, x_{n_2}, \dots$ that converges:

$$x_{n_j} \to c \in [a,b] \text{ as } j \to \infty.$$

By continuity,

$$f(c) = f(\lim_{j\to\infty} x_{n_j}) = \lim_{j\to\infty} f(x_{n_j}),$$

so that the sequence $(f(x_{n_j}))_{j\in\mathbb{N}}$ converges. It follows (check this) that the sequence $(|f(x_{n_j})|)_{j\in\mathbb{N}}$ converges, and is therefore bounded. This is a contradiction since each $|f(x_{n_j})|$ exceeds n_j.

It follows that f must be bounded on $[a,b]$. □

Theorem 5.2 says that a continuous function on a closed bounded interval is bounded. In fact more is true: a continuous function on a closed bounded interval *attains its bounds*. Without some assumptions, this can fail even for bounded functions: for example the function $f(x) = x$ on the set $(0,1)$ is bounded and continuous, but does not attain its bounds.

Theorem 5.3. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there are points $\xi, \eta \in [a,b]$ such that*

$$f(\xi) = \sup\{f(x) \mid x \in [a,b]\},$$

and

$$f(\eta) = \inf\{f(x) \mid x \in [a,b]\}.$$

Proof. We shall prove one of these – the other follows at once (Why?). Let

$$M = \sup\{f(x) \mid x \in [a, b]\}.$$

Notice that M exists because of Theorem 5.2. Now by the definition of a sup, there must be values of $f(x)$ arbitrarily close to M: for each $n \in \mathbb{N}$ we may find $x_n \in [a, b]$ such that

$$M - \tfrac{1}{n} < f(x_n) \leq M.$$

By the Bolzano–Weierstrass Theorem, we may choose a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ that converges to $\xi \in [a, b]$ say. Then by continuity of f,

$$f(\xi) = f(\lim_{j\to\infty} x_{n_j}) = \lim_{j\to\infty} f(x_{n_j}) = M,$$

by the Sandwich Theorem. □

Before proving the Intermediate–Value Theorem, notice the following consequences of Definition 4.2.

(1) If $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) < \lambda$ then for some $\delta > 0$, $f(x) < \lambda$ for all x in $[a, a + \delta)$.
(2) If $f : [a, b] \to \mathbb{R}$ is continuous, and $f(b) > \lambda$ then for some $\delta > 0$, $f(x) > \lambda$ for all x in $(b - \delta, b]$.

Theorem 5.4. [INTERMEDIATE–VALUE THEOREM] *Suppose that* $f : [a, b] \to \mathbb{R}$ *is continuous, and that* $f(a) < \lambda < f(b)$. *Then there is a point* $\xi \in (a, b)$ *for which* $f(\xi) = \lambda$.

Proof. Define a set A as follows:

$$A = \{x \mid a \leq x \leq b, \text{ and } f < \lambda \text{ on the interval } [a, x].\}$$

It is clear that A is non–empty since $a \in A$. Also, b is an upper bound for A. It follows by the Axiom of Completeness that A has a least upper bound ξ, with $a \leq \xi \leq b$. By the remark above (and the homework), we know that there is a $\delta > 0$ such that $[a, a + \delta) \subset A$ and $b - \delta$ is an upper bound for A, so in fact $a < \xi < b$. This is used below – check that you understand where.

To prove that $f(\xi) = \lambda$, we eliminate the two other possibilities $f(\xi) < \lambda$ and $f(\xi) > \lambda$.

Suppose first that $f(\xi) < \lambda$. Then if $\epsilon = \lambda - f(\xi)$, let $\delta > 0$ be chosen as in Definition 4.1 so that

$$|x - \xi| < \delta \implies |f(x) - f(\xi)| < \epsilon.$$

Then $x = \xi + \frac{1}{2}\delta$ has $f(x) < f(\xi) + \epsilon = \lambda$, so $\xi + \frac{1}{2}\delta$ lies in A. This contradicts the fact that ξ is an upper bound of A.

Suppose, on the other hand, that $f(\xi) > \lambda$. Again,let $\epsilon = f(\xi) - \lambda$ and choose $\delta > 0$ to have

$$|x - \xi| < \delta \implies |f(x) - f(\xi)| < \epsilon.$$

Then $x = \xi - \frac{1}{2}\delta$ has $f(x) > f(\xi) + \epsilon = \lambda$, so $\xi - \frac{1}{2}\delta$ is an upper bound for A, contradicting the fact that ξ is the *least* upper bound of A.

The only remaining possibility is that $f(\xi) = \lambda$. □

Examples 5.5. (1) Let $f(x) = x^2 - 2$. Use the IVT to show that there is a real number whose square is 2.
(2) Prove that any cubic polynomial has a real zero.
(3) Let $f : [0,1] \to [0,1]$ be any continuous function. Then f has a *fixed point*, that is a point c with $f(c) = c$.

This is proved by applying the IVT to the continuous function

$$g(x) = f(x) - x.$$

(4) Prove that the equation

$$\frac{x^2}{x^2-1} + \frac{2x^2+3}{x^2-4} = 5$$

has four real solutions.

Things you should have learned in this section
(1) How to prove the Bolzano–Weierstrass theorem.
(2) How to apply it to prove results about continuous functions.
(3) How to prove and use the IVT.

TBW/1A22

6. DIFFERENTIABILITY

Objectives

(1) To make precise the property of differentiability at a point.
(2) To use known methods on limits to prove that simple functions are differentiable.
(3) To establish an algebra of differentiable functions.

We saw in Section 1 how to find the slope or gradient of the chord joining a pair of points $(x, f(x))$ and $(x+h, f(x+h))$ on the graph of a curve $y = f(x)$: the slope is

$$\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}.$$

Definition 6.1. Let f be a function defined on some neighbourhood of c. Then f is *differentiable at* c if the limit

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

exists. If the limit exists, it is called the *derivative of* f *at* c, and is denoted $f'(c)$.

Notice that the definition may also be written as

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}.$$

Of course the quantity h must be allowed to be negative or positive: if h is restricted to be positive we arrive at the definition of f being *differentiable from the right.*

A function which is differentiable at every point of a set is said to be differentiable on the set, and a function that is differentiable at each point of its domain is simply called differentiable.

Examples 6.2. (1) Prove that $f(x) = x^2$ is differentiable.
Proof: Notice that

$$\frac{f(x+h)-f(x)}{h} = \frac{x^2+2xh+h^2-x^2}{h} = 2x+h,$$

from which it is clear that $f'(x)$ exists and is equal to $2x$.
(2) We don't really have a definition of the function $\sin(x)$, but this function is so important that we shall make a provisional definition. Define (for now) a function $\sin : \mathbb{R} \to \mathbb{R}$ to be the signed length of the opposite edge of a right triangle with hypotenuse length 1 and angle x in radians.

The first step is to show that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

To see that this is plausible, notice that the perimeter of a regular n–gon inscribed in a circle of radius 1 is given by $2n\sin(\pi/n)$. As $n \to \infty$, this perimeter should converge to 2π. It follows that

$$2n\sin(\pi/n) \to 2\pi,$$

so if we let $x = \pi/n$ we have as $x \to 0$,

$$(2\pi)\frac{\sin(x)}{x} = 2\pi.$$

We also define $\cos(x)$ via the relation $\sin^2(x)+\cos^2(x) = 1$ (and appropriate sign choices). Both $\sin(x)$ and $\cos(x)$ are continuous functions (this can be shown from the geometric definition), and the usual double– and half– angle formulæ may be proved geometrically.

Now

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{2\cos(x+\frac{h}{2}))\sin(\frac{h}{2})}{h} = \cos(x+\tfrac{h}{2})\frac{\sin(\frac{h}{2})}{\frac{h}{2}},$$

and as $h \to 0$, $\cos(x+\frac{h}{2}) \to \cos(x)$ and $\frac{\sin(h/2)}{h/2} \to 1$. It follows that $\sin(x)$ is everywhere differentiable and $\sin'(x) = \cos(x)$.

Theorem 6.3. *If f is differentiable at c then f is continuous at c.*

Proof. The function f is defined in some neighbourhood of c, and for values of x in that neighbourhood,

$$\frac{f(x)-f(c)}{x-c}$$

converges to $f'(c)$ as $x \to c$. It follows that

$$f(x)-f(c) = \frac{f(x)-f(c)}{x-c}\cdot(x-c)$$

converges to $f'(c)\times 0 = 0$, so f is continuous at c. □

Remarks 6.4. (1) This says that differentiability is a stronger property than continuity. In Section 1 we saw that the function $x \mapsto |x|$ is continuous everywhere but is not differentiable at $x = 0$. Weierstrass extended this example to exhibit a function that is continuous at every point but differentiable at no points.
(2) Any function that is not continuous at a point cannot be differentiable there.

Theorem 6.5. *If f and g are differentiable at c then*

1. $(f+\lambda g)'(c)$ *exists and equals* $f'(c)+\lambda g'(c)$,
2. $(fg)'(c)$ *exists and equals* $f(c)g'(c)+f'(c)g(c)$, *and*
3. $(f/g)'(c)$ *exists if* $g(c) \neq 0$ *and equals* $\frac{g(c)f'(c)-f(c)g'(c)}{[g(c)]^2}$.

Proof. (1) and (3) are exercises. For (2), f and g are defined in some neighbourhood of c so for small values of h,

$$\frac{f(c+h)g(c+h)-f(c)g(c)}{h} = f(c+h)\frac{g(c+h)-g(c)}{h} + g(c)\frac{f(c+h)-f(c)}{h}$$

$$\longrightarrow f(c)g'(c) + f'(c)g(c)$$

as $h \to 0$ since $f'(c)$, $g'(c)$ exist and f is continuous at $x = c$. ☐

Higher–order derivatives $f''(x)$ and so on are defined in the obvious way: $f''(c)$, if it exists, is $(f')'(c)$. Notice that, for example, if f has a fourth derivative $f^{(iv)}$ then f', f'' and f''' must exist – and all of them must be continuous.

Things you should have learned in this section

(1) The definition of differentiability.

(2) How to prove from the definition that simple functions are differentiable.

7. Applications

Objectives

(1) To understand the relationship between maxima and minima and the derivative for differentiable functions.
(2) To prove Rolle's theorem and deduce the Mean–Value Theorem.
(3) To prove the Cauchy Mean–Value Theorem and deduce l'Hôpital's rule.
(4) To understand that the MVT is a simple form of Taylor's theorem.
(5) To see a form of Taylor's Theorem with remainder. Proof included in notes but non–examinable.
(6) To see how Taylor's theorem may be used to analyse maxima and minima when higher–order derivatives vanish.

Continuity (via the IVT) allowed us to say when graphs of functions cross each other. Differentiation will tell us when graphs of functions are rising or falling.

A function $f : D \to \mathbb{R}$ is

(1) *increasing* on $S \subset D$ if $f(x) \leq f(y)$ whenever $x, y \in S$ have $x < y$, and
(2) *strictly increasing* on $S \subset D$ if $f(x) < f(y)$ whenever $x, y \in S$ have $x < y$.

(You can work out the meaning of *decreasing* and *strictly decreasing* yourself.) A function is called *(strictly) monotone* if it is either (strictly) increasing or (strictly) decreasing.

Definition 7.1. Let f be defined in some neighbourhood of c. Then f has a *local maximum* (resp. *minimum)* at c if $f(x) \leq f(c)$ (resp. $f(x) \geq f(c)$) for all x in some neighbourhood of c.

Example 7.2. The function $f(x) = x^2 + 4x + 1$ is strictly decreasing on the interval $(-\infty, -2]$, strictly increasing on $[-2, \infty)$ and has a local minimum at 2.

Theorem 7.3. *Suppose f is differentiable at c and has a local maximum or minimum at c. Then $f'(c) = 0$.*

Warning. As usual, beware of the converse: the function $x \mapsto x^3$ has zero derivative at $x = 0$ but has no local maxima or minima.

Proof. Suppose f has a local maximum at c. Then for $h > 0$ and small,

$$\frac{f(x+h) - f(x)}{h} \leq 0$$

while for $h < 0$ and small,

$$\frac{f(x+h) - f(x)}{h} \geq 0.$$

We know f that is differentiable at c, so

$$f'(c) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \leq 0,$$

and

$$f'(c) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \geq 0,$$

which shows that $f'(c) = 0$. ☐

A point c at which $f'(c) = 0$ is called a *stationary point* of f. The next Theorem says that a differentiable function that goes up and then down (or down and then up) must have a stationary point.

Theorem 7.4. [Rolle's Theorem] *If $f : [a,b] \to \mathbb{R}$ is continuous, f is differentiable on (a,b), and $f(a) = f(b)$, then for some $c \in (a,b)$, $f'(c) = 0$.*

Proof. By Theorem 5.3 and continuity on $[a,b]$, there are points ξ, η such that

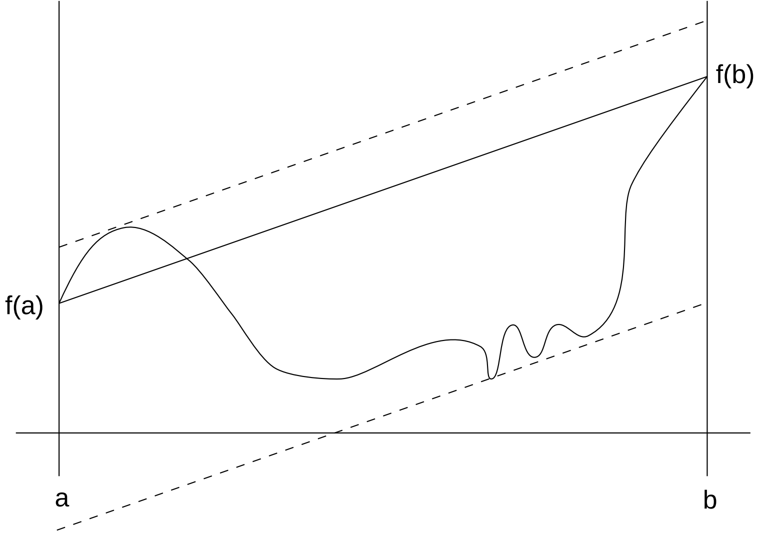
$$f(\xi) = \sup\{f(x) \mid x \in [a,b]\},$$

and

$$f(\eta) = \inf\{f(x) \mid x \in [a,b]\}.$$

If either ξ or η lies in (a,b) we are done by Theorem 7.3. If not, then f is constant and so any point $c \in (a,b)$ will do. ☐

More generally, a differentiable function whose average (or mean) slope over some interval is λ must have a point whose derivative is λ.



Theorem 7.5. [Mean Value Theorem] *If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a point $c \in (a,b)$ with*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Let $g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$. Check that g satisfies the hypotheses of Rolle's Theorem. ☐

Corollary 7.6. *If $f'(x) = 0$ on (a, b) and f is continuous on $[a, b]$, then f is constant on $[a, b]$.*

Proof. For any $x \in (a, b]$, f is continuous on $[a, x]$ and differentiable on (a, x), so by the MVT there is a $c \in (a, x)$ such that $f'(c) = 0 = \frac{f(x)-f(a)}{x-a}$, so $f(x) = f(a)$ for all $x \in (a, b]$. ☐

Examples 7.7. (1) Deduce that if f and g have the same derivative on $[a, b]$, then they differ by a constant.
(2) The same idea we used in Corollary 7.6 gives other useful results. For example, if $f'(x) \geq 0$ on (a, b) and f is continuous on $[a, b]$, then f is increasing on $[a, b]$
(3) In (2) – and on the homework – we have for example the result that a differentiable function whose derivative is positive is strictly increasing. The converse does not hold of course: $x \mapsto x^3$ is strictly increasing everywhere but does have a stationary point.

Theorem 7.8. [Cauchy Mean–Value Theorem] *Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b), with $g'(x) \neq 0$ for $x \in (a, b)$. Then there is a point $c \in (a, b)$ for which*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Notice that an argument is needed: if the MVT is applied to the two functions f, g, we will get two different points c, d with

$$\frac{f'(c)}{g'(d)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

First apply the MVT to g to see that $g(b) - g(a) \neq 0$. Next, define

$$\phi(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x),$$

and notice that ϕ satisfies the hyptheses of the MVT. Hence there is a point $c \in (a, b)$ such that

$$\phi(b) - \phi(a) = (b - a)\phi'(c).$$

Now $\phi(b) - \phi(a) = 0$, so

$$[g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c) = 0,$$

which gives the result since $g(b) - g(a) \neq 0$. ☐

In Section 2 we were not able to compute expressions of the form $\lim_{x \to a} \frac{f(x)}{g(x}$ when $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$. The Cauchy Mean Value Theorem allows us to say more about this in a special case.

Theorem 7.9. [L'Hôpital's Rule] *Suppose that*

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0,$$

and also suppose that $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ *exists. Then* $\lim_{x\to a}\frac{f(x)}{g(x)}$ *exists, and*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof. The hypothesis that $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists implies that

(1) $f'(x)$ and $g'(x)$ exist on the punctured neighbourhood $(a-\delta, a)\cup(a, a+\delta)$ for some $\delta > 0$;
(2) in this punctured neighbourhood of a, $g'(x) \neq 0$.

Now *a priori* f and g need not be defined at a; define $f(a) = g(a) = 0$ so that f and g become continuous at a. Now consider the interval $[a, x]$ for some $x \in (a, a+\delta)$. Applying the Mean Value Theorem to g, we see that $g(x) \neq 0$ (Why?). Applying the Cauchy Mean Value Theorem to f and g, there is a number $\alpha_x \in (a, x)$ such that

$$[f(x) - 0]g'(\alpha_x) = [g(x) - 0]f'(\alpha_x),$$

so

$$\frac{f(x)}{g(x)} = \frac{f'(\alpha_x)}{g'(\alpha_x)}.$$

Now as x approaches a, α_x approaches a; since $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists it follows that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(\alpha_x)}{g'(\alpha_x)} = \lim_{y\to a}\frac{f'(y)}{g'(y)}.$$

□

Notice that the MVT is a special case of Taylor's Theorem: let $a = 0$, $b = x$, and then it says that if f is continuous on $[0, x]$ and differentiable on $(0, x)$, then

$$f(x) = f(0) + x \cdot f'(\xi)$$

for some $\xi \in (0, x)$.

Theorem 7.10. [Taylor's Theorem] *Suppose* $f, f', f'', \ldots, f^{(N)}$ *are continuous on* $[a, b]$*, and* $f^{(n+1)}$ *exists on* (a, b)*. Then, for each* $x \in (a, b)$ *there is a* $\theta \in (a, x)$ *such that*

$$f(x) = \sum_{n=0}^{N}\frac{f^{(n)}(a)}{n!}(x-a)^n + R_N$$

where

$$R_N = \frac{f^{(N+1)}(\theta)}{(N+1)!}(x-a)^{N+1}.$$

Proof. Define $R_N(x) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!}(x-a)^n$. Then $R_N(x)$ is differentiable.

Now consider the function

$$\Phi(y) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(y)}{n!}(x-y)^n - R_N(x)\frac{(x-y)^{N+1}}{(x-a)^{N+1}}.$$

Notice that $\Phi(x) = \Phi(a) = 0$, $\Phi(y)$ is continuous on $[a, x]$, and Φ' exists on (a, x). By Rolle's Theorem, there is a $\theta \in (a, x)$ such that $\Phi'(\theta) = 0$. On the other hand,

$$\begin{aligned}\Phi'(y) = &- \sum_{n=0}^{N} \frac{f^{(n+1)}(y)}{n!}(x-y)^n + \sum_{n=1}^{N} \frac{f^{(n)}(y)}{n!} \cdot n \cdot (x-y)^{n-1} \\ &+ R_N(x) \cdot \frac{(N+1)(x-y)^N}{(x-a)^{N+1}} \\ = &\frac{f^{(N)}(y)}{N!}(x-y)^N + R_N(x) \cdot \frac{(N+1)(x-y)^N}{(x-a)^{N+1}}.\end{aligned}$$

Substitute $y = \theta$ to obtain

$$0 = \Phi'(\theta) = -\frac{f^{(N+1)}(\theta)}{N!}(x-\theta)^N + R_N(x) \cdot \frac{(N+1)(x-\theta)^N}{(x-a)^{N+1}},$$

so

$$R_N(x) = \frac{f^{(N+1)}(\theta)}{(N+1)!}(x-a)^{N+1}.$$

□

Examples 7.11. (1) Let $f(x) = e^x$. Since $f'(x) = e^x$, for $0 < x$ there is a $\theta \in (0, x)$ such that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \frac{e^\theta x^{N+1}}{(N+1)!}.$$

For any fixed value of θ,

$$\frac{\theta^{N+1}}{(N+1)!} \to 0$$

as $N \to \infty$ so there is an expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

covergent for all x[1].

(2) Let $f(x) = \log(1+x)$. Then $f'(x) = \frac{1}{1+x}$, $f''(x) = \frac{-1}{(1+x)^2}$ and so on. So for any $x > 0$, there is a $\theta \in (0, x)$ with

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{N-1}x^N}{N} + \frac{(-1)^N x^{N+1}}{(1+\theta)^{N+1}N+1}.$$

We can only be sure that the remainder term will converge to zero if $|\theta| < 1$, so we have the convergent series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

valid for $|x| < 1$.

---

[1] Treat this example with some caution: it is perfectly correct, but the function $x \mapsto e^x$ is really *defined* to be this Taylor series.

Theorem 7.12. *Let f be differentiable $(n+1)$ times on $[a-h, a+h]$, and assume that $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$. Then*

1. *f has a local minimum at a if $f^{(n)}(a) > 0$ and n is even.*
2. *f has a local maximum at a if $f^{(n)}(a) < 0$ and n is even.*
3. *f has a point of inflection at a if n is odd.*

Proof. What determines the sign of the remainder term in Taylor's Theorem? □

Things you should have learned in this section

(1) Statement and proof of Rolle's Theorem, Cauchy MVT, MVT.
(2) How to use the MVT to relate the derivative to the behaviour of a function.
(3) Exact statement of l'Hôpital's rule and how to use it to find indeterminate limits.
(4) Exact statement of Taylor's Theorem with remainder and how to use it to show that standard Taylor series converge.
(5) How to use Taylor's theorem to prove results about maxima and minima.

TBW/1A22

8. The Riemann Integral

Objectives
(1) To define rigorously the definite integral.
(2) To find methods for proving that functions are integrable.
(3) To see examples of integration from first principles.

In this section we will give a rigorous definition of the meaning of expressions like

$$\int_a^b f(x)dx.$$

The basic idea has already been described in Section 1: the definite integral above represents the area under the graph of f from $x = a$ to $x = b$, which may be approximated by rectangular strips. The technical questions to be dealt with are the following. First, when this process works, does it matter how the strips are chosen? Second, for which functions does this process of approximation converge?

Throughout this section, we shall be dealing with a bounded function $f : [a, b] \to \mathbb{R}$.

Definition 8.1. A *partition*, $\mathcal{P}$, of the interval $[a, b]$, is a finite set of points $\{x_0, x_1, \ldots, x_k\}$ satisfying

$$a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b.$$

Definition 8.2. If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $[a, b]$ and $\mathcal{P} \supset \mathcal{Q}$, then $\mathcal{P}$ is a *refinement* of $\mathcal{Q}$.

Corresponding to a partition $\mathcal{P}$ are the *upper and lower sums for f over $\mathcal{P}$*,

$$U(\mathcal{P}) = \sum_{i=1}^{k} M_i(x_i - x_{i-1}),$$

$$L(\mathcal{P}) = \sum_{i=1}^{k} m_i(x_i - x_{i-1})$$

where $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$.

Notice that M_i and m_i exist since f is bounded. If $M = \sup\{f(x) \mid x \in [a, b]\}$ and $m = \inf\{f(x) \mid x \in [a, b]\}$ then

$$m(b - a) \leq L(\mathcal{P}) \leq U(\mathcal{P}) \leq M(b - a).$$

Let Φ denote the set of all partitions of $[a, b]$. Then, by the above,

$$\{U(\mathcal{P}) \mid \mathcal{P} \in \Phi\}$$

is non–empty and bounded, so

$$U = \inf\{U(\mathcal{P}) \mid \mathcal{P} \in \Phi\}$$

exists. Similarly,

$$L = \sup\{L(\mathcal{P}) \mid \mathcal{P} \in \Phi\}$$

exists.

Definition 8.3. If $L = U$ then f is *Riemann integrable over* $[a, b]$, and the common value of U and L is the *Riemann integral of f over* $[a, b]$, denoted $\int_a^b f$ or $\int_a^b f(x)dx$.

Lemma 8.4. *If $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then*

$$L(\mathcal{P}) \leq L(\mathcal{Q}) \leq U(\mathcal{Q}) \leq U(\mathcal{P}).$$

Given two partitions $\mathcal{P}$ and $\mathcal{Q}$, their union $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$ is called the *common refinement* of $\mathcal{P}$ and $\mathcal{Q}$.

Lemma 8.5. *If $\mathcal{P}$ and $\mathcal{Q}$ are any partitions of* $[a, b]$ *then*

$$L(\mathcal{P}) \leq U(\mathcal{Q})$$

and so

$$L(\mathcal{P}) \leq L \leq U \leq U(\mathcal{Q}).$$

Proof. Let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$ denote the common refinement of $\mathcal{P}$ and $\mathcal{Q}$. Then by Lemma 8.4,

$$U(\mathcal{R}) \leq U(\mathcal{Q}) \text{ and } L(\mathcal{P}) \leq L(\mathcal{R}).$$

However, $L(\mathcal{R}) \leq U(\mathcal{R})$ and so $L(\mathcal{P}) \leq U(\mathcal{Q})$.

Since $L(\mathcal{P}) \leq U(\mathcal{Q})$ for all $\mathcal{P}, \mathcal{Q} \in \Phi$, we obtain for fixed $\mathcal{P}$ that

$$L(\mathcal{P}) \leq \inf\{U(\mathcal{Q}) \mid \mathcal{Q} \in \Phi\} \leq U,$$

and so $L(\mathcal{P}) \leq L \leq U \leq U(\mathcal{Q})$. □

Example 8.6. Define $f(x) = c$ for $x \in [a, b]$. Prove that $\int_a^b f = c(b - a)$.
Let $\mathcal{P} = \{x_0, x_1, \ldots, x_k\}$ be any partition of $[a, b]$. It is clear that $L(\mathcal{P}) = U(\mathcal{P}) = c(b - a)$, and since $\mathcal{P}$ was arbitrary, this shows that $\int_a^b f = c(b - a)$.

Several of the main theorems on integration will not be proved in this course – but for most of them you should be able to come up with a proof.

Theorem 8.7. [DARBOUX'S CRITERION] *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if, for every $\epsilon > 0$ there is a partition $\mathcal{P}$ for which $U(\mathcal{P}) - L(\mathcal{P}) < \epsilon$.*

Proof. Suppose first that for every $\epsilon > 0$ there is a partition $\mathcal{P}$ with

$$U(\mathcal{P}) - L(\mathcal{P}) < \epsilon.$$

Since

$$\inf\{U(\mathcal{Q}) \mid \mathcal{Q} \in \Phi\} \leq U(\mathcal{P})$$
$$\sup\{L(\mathcal{Q}) \mid \mathcal{Q} \in \Phi\} \geq L(\mathcal{P})$$

it follows that

$$\inf\{U(\mathcal{Q}) \mid \mathcal{Q} \in \Phi\} - \sup\{L(\mathcal{Q}) \mid \mathcal{Q} \in \Phi\} < \epsilon.$$

Since this holds for all $\epsilon > 0$, it follows that

$$\inf\{U(\mathcal{Q}) \mid \mathcal{Q} \in \Phi\} = \sup\{L(\mathcal{Q}) \mid \mathcal{Q} \in \Phi\};$$

so by definition f is Riemann integrable on $[a, b]$.

Conversely, if f is integrable then

$$\inf\{U(\mathcal{Q}) \mid \mathcal{Q} \in \Phi\} = \sup\{L(\mathcal{Q}) \mid \mathcal{Q} \in \Phi\}.$$

This means that for each $\epsilon > 0$ there are partitions $\mathcal{Q}$, $\mathcal{R}$ such that

$$U(\mathcal{Q}) - L(\mathcal{R}) < \epsilon.$$

Let $\mathcal{P} = \mathcal{Q} \cup \mathcal{R}$ be the common refinement of $\mathcal{Q}$ and $\mathcal{R}$. Then by Lemma 8.4,

$$\begin{aligned} U(\mathcal{P}) &\leq U(\mathcal{Q}) \\ L(\mathcal{P}) &\geq L(\mathcal{R}), \end{aligned}$$

so

$$U(\mathcal{P}) - L(\mathcal{P}) \leq U(\mathcal{Q}) - L(\mathcal{R}) < \epsilon.$$

□

The next Theorem requires a notion (*uniform continuity*) beyond this course and is therefore left without proof.

Theorem 8.8. *If $f : [a, b] \to \mathbb{R}$ is continuous or monotone then it is Riemann integrable.*

To end this section, we shall do the example from Section 1 again, but this time justifying everything.

Example 8.9. Let $f(x) = x$. From the definition, find the Riemann integral $\int_0^1 f$. First, by Theorem 8.8 the integral exists – all we need to do is find the value. Let $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$. By the calculation in Section 1,

$$U(\mathcal{P}_n) = \frac{1}{2} + \frac{1}{2n} \text{ and } L(\mathcal{P}_n) = \frac{1}{2} - \frac{1}{2n}.$$

Now apply Lemma 8.5 to deduce that

$$\frac{1}{2} - \frac{1}{2n} = L(\mathcal{P}_n) \leq L \leq U \leq U(\mathcal{P}_n) = \frac{1}{2} + \frac{1}{2n},$$

and so

$$\frac{1}{2} - \frac{1}{2n} \leq \int_0^1 f \leq \frac{1}{2} + \frac{1}{2n}.$$

Letting n tend to infinity gives the result by the Sandwich Theorem.

Notice in this example how Lemma 8.5 is used to let us pass from convenient special partitions to statements that apply to all partitions.

Things you should have learned in this section

(1) The definition of the Riemann integral.
(2) Darboux's criterion and how to use it.
(3) How to compute simple definite integrals from first principles.

TBW/1A22

9. General Theorems on Integration

Objectives

(1) To determine basic general properties of the definite integral.

(2) To relate properties of continuous functions to properties of their integrals.

Let f and g be bounded functions on $[a, b]$.

Theorem 9.1. *If f and g are Riemann integrable on $[a, b]$, then*

1. *for $c \in \mathbb{R}$, $cf(x)$ is also Riemann integrable and $\int_a^b cf = c \int_a^b f$.*
2. *$f + g$ is Riemann integrable and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.*
3. *$f \times g$ is Riemann integrable.*
4. *if $f \geq 0$ on $[a, b]$, then $\int_a^b f \geq 0$.*
5. *$|f|$ is Riemann integrable, and $|\int_a^b f| \leq \int_a^b |f|$.*

Proof. (2), (4) and (5) are easy – prove them as an exercise. You can look up (3) in one of the references.

Here we shall prove the confusing case in (1): if $c \geq 0$ then (1) is easy, so assume that $c < 0$. Let $\mathcal{P} = \{x_0, \ldots, x_k\}$ be a partition of $[a, b]$. Then

$$M_i(cf) = \sup\{cf(x) \mid x \in [x_{i-1}, x_i]\} = c \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

so that $U(\mathcal{P}) = cL(\mathcal{P})$. It follows that $U(cf) = cL(f)$. A similar argument gives $L(cf) = cU(f)$, which gives the result. $\square$

Remark 9.2. So far we have defined $\int_a^b f$ for $a < b$. If we define $\int_a^b f = -\int_b^a$, and $\int_a^a f$ to be 0 if f is defined at a then

$$\int_a^b f + \int_b^c f = \int_a^c f$$

whenever either side of the equality is defined.

Example 9.3. Let f be a continuous function on $[a, b]$ with the property that $f(x) \geq 0$ for all $x \in [a, b]$ and $f(c) > 0$ for some $c \in [a, b]$. Prove that $\int_a^b f > 0$. Solution. The point here is to show the integral is *positive* – it is clear that the integral is *non–negative*. Assume for now that $c \in (a, b)$ (the case $c = a$ or $c = b$ is similar), and let $\epsilon = f(c)/2 > 0$. By continuity, there is a $\delta > 0$ with the property that

$$|x - c| < \delta \implies x \in [a, b] \text{ and } |f(x) - f(c)| < \epsilon,$$

which implies that $f(x) > f(c) - \epsilon > f(c)/2$. (Draw a picture).

It follows that

$$\int_{c-\delta}^{c+\delta} f \geq \frac{f(c)}{2} \times (2\delta) > 0.$$

On the other hand

$$\int_a^{c-\delta} f \geq 0$$

and

$$\int_{c+\delta}^{b} f \geq 0,$$

so we are done.

Things you should have learned in this section

(1) General properties of integration (linearity, positivity and so on).
(2) Estimates for integrals.

TBW/1A22

10. The Fundamental Theorem(s) of Calculus

Objectives

(1) To prove both forms of the fundamental theorem of calculus.
(2) To apply the fundamental theorem.

Theorem 10.1. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable, and define a function $\phi : [a,b] \to \mathbb{R}$ by*

$$\phi(x) = \int_a^x f.$$

Then ϕ is continuous.

Proof. Let x be a point in $[a,b]$ and let $h \neq 0$ be chosen with the property that $x+h \in [a,b]$. Then

$$\phi(x+h) - \phi(x) = \int_x^{x+h} f.$$

(Notice that if $x = b$, then h is automatically negative).

Now f is bounded on $[a,b]$, so there is a K such that $|f(t)| \leq K$ for all $t \in [a,b]$. It follows that

$$|\phi(x+h) - \phi(x)| = |\int_x^{x+h} f| \leq K|h| \to 0$$

as $h \to 0$. ☐

Theorem 10.2. [FIRST FUNDAMENTAL THEOREM OF CALCULUS] *If $f : [a,b] \to \mathbb{R}$ is integrable over $[a,b]$ and continuous at the point $c \in [a,b]$ then $\phi(x) = \int_a^x f$ is differentiable at c. Moreover, $\phi'(c) = f(c)$.*

Notice that c might be a or b, in which case the theorem is true with the obvious left– or right– sided notion of differentiability.

Proof. Let $h \neq 0$ be chosen with $c+h \in [a,b]$. Then

$$\begin{aligned}\frac{\phi(c+h) - \phi(c)}{h} - f(c) &= \frac{1}{h}\left(\int_a^{a+h} f - \int_a^c f\right) - f(c) \\ &= \frac{1}{h}\int_c^{c+h} f - f(c) \\ &= \frac{1}{h}\int_c^{c+h} [f(t) - f(c)]dt. \qquad (1)\end{aligned}$$

Let $\epsilon > 0$ be given. Since f is continuous at $x = c$, there is a $\delta > 0$ for which

$$t \in (c-\delta, c+\delta) \cap [a,b] \implies |f(t) - f(c)| < \epsilon. \qquad (2)$$

It follows that for $|h| < \delta$, by (1)

$$\begin{aligned}\left|\frac{\phi(c+h)-\phi(c)}{h} - f(c)\right| &= \frac{1}{|h|}\left|\int_c^{c+h} [f(t)-f(c)]dt\right| \\ &\leq \frac{1}{|h|}\left|\int_c^{c+h} |f(t)-f(c)|dt\right| \\ &\leq \frac{1}{|h|}\left|\int_c^{c+h} \epsilon dt\right| = \epsilon,\end{aligned}$$

so $\phi'(c)$ exists and equals $f(c)$. □

Theorem 10.3. [SECOND FUNDAMENTAL THEOREM OF CALCULUS] *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$. Then there exists a function $F : [a, b] \to \mathbb{R}$ such that $F'(x)$ exists and equals $f(x)$ on $[a, b]$, and*

$$\int_a^b f = F(b) - F(a). \tag{1}$$

Moreover, if F is any function such that $F'(x) = f(x)$ on $[a, b]$ then (1) holds.

Proof. Since f is continuous on $[a, b]$, it is Riemann integrable over $[a, b]$. Define as usual $\phi(x) = \int_a^x f$ for $x \in [a, b]$.

Since f is continuous, $\phi'(x)$ exists and equals $f(x)$ by Theorem 10.2. Also, by definition of ϕ,

$$\int_a^b f = \phi(b) = \phi(b) - \phi(a).$$

That is, for the first part of the Theorem we may choose F to be ϕ.

Now let F be any function with the property that $F'(x) = f(x)$ on $[a, b]$. Since F is differentiable, it is certainly continuous on $[a, b]$, as is ϕ. So $G(x) = F(x) - \phi(x)$ is continuous on $[a, b]$, and $G'(x) = F'(x) - \phi'(x) = 0$, so $G(x) = C$ for some constant C. Then $F(x) = \phi(x) + C$, and so $F(a) = C$ since $\phi(a) = 0$. Hence $\int_a^b f = \phi(b) = F(b) - C = F(b) - F(a)$. □

Things you should have learned in this section

(1) Statement and proof of both versions of the fundamental theorem of calculus.

11. "Elementary" Calculus

Objectives

(1) To review some elementary symbolic integration.

Now that the basic theory is set up, we can indicate some extensions of integration, and some techniques for computing integrals and derivatives (it is sometimes useful to be able to do this).

Improper Integrals

An integral is said to be *improper* if it involves infinity, either explicitly:

$$J_p = \int_1^\infty \frac{1}{x^p} dx$$

or implicitly:

$$I_p = \int_0^1 \frac{1}{x^p} dx.$$

In either case, the approach is the same: the "infinity" is apporached as a limit.

Example 11.1. The improper integral $\int_1^\infty f$ exists if the limit

$$\lim_{R\to\infty} \int_1^R f$$

exists, and in this case is defined to be this limit.

Example 11.2. Similarly, if f is undefined at 0 say but is defined elsewhere, then $\int_0^1 f$ is defined to be

$$\lim_{r\to 0} \int_r^1 f$$

if this limit exists.

We have seen this already in 1A11, where improper integrals were used in the Integral Test for convergence of series.

Logarithms

For $x > 0$, define a function by

$$\log(x) = \ln(x) = \int_1^x \tfrac{1}{t} dt.$$

By the Fundamental Theorem, $\log'(x) = \frac{1}{x}$. Choose a number $y > 0$ and let

$$f(x) = \log(xy).$$

Then $f'(x) = \log'(xy) \cdot y = \frac{1}{xy} \cdot y = \frac{1}{x}$. Thus $f' = \log'$. This means that there is a constant c such that

$$f(x) = log(x) + c \text{ for all } x > 0.$$

When $x = 1$ we see that $c = \log(y)$, so

$$\log(xy) = \log(x) + \log(y) \text{ for all } x, y > 0.$$

It follows easily that for $n \in \mathbb{N}$ and $x, y > 0$,

$$\log(x^n) = n \log(x)$$

and

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y).$$

Since $\log(2^n) = n\log(2)$ and $\log(2^{-n}) = -n\log(2)$, log is unbounded below on $(0, 1)$ and is unbounded above on $[1, \infty)$. It is also strictly increasing, so log is a bijection from $(0, \infty)$ to $\mathbb{R}$.

Definition. The inverse of log is the exponential function, $\exp : \mathbb{R} \to (0, \infty)$.

Notice that

$$\begin{aligned}\exp'(x) = (\log^{-1})'(x) &= \frac{1}{\log'(\log^{-1}(x))} \\ &= \frac{1}{\frac{1}{\log^{-1}(x)}} \\ &= \log^{-1}(x) = \exp(x).\end{aligned}$$

Other properties follow from the properties of log; for example, $\exp(x+y) = \exp(x)\exp(y)$.

INTEGRATION TECHNIQUES

It is sometimes important to be able to evaluate integrals in elementary terms. The first step is to notice that the fundamental theorem shows that every derivative we compute yields an integral: if we observe that

$$F(x) = x\arctan(x) - \frac{\log(1+x^2)}{2}$$

has

$$F'(x) = \arctan(x),$$

then we may deduce that

$$\int_a^b \arctan(x)dx = x\arctan(x) - \frac{\log(1+x^2)}{2}|_a^b.$$

For brevity, we will deal with *primitives* or *indefinite integrals*, that is any function whose derivative is equal to the integrand.

[1] INTEGRATION BY PARTS. If f' and g' are continuous, then

$$\int fg' = fg - \int f'g.$$

Example 11.3. (1) Let $g'(x) = \frac{1}{x}$ and $f(x) = \log x$. Then

$$\int (1/x) \log x dx = \log x \cdot \log x - \int (1/x) \log x dx,$$

so

$$\int \tfrac{1}{x} \log x dx = \tfrac{(\log x)^2}{2}.$$

(2) A simple calculation by parts gives $\int \log x dx = x(\log x) - x$. Let $f(x) = \log x$ and $g'(x) = \log x$. Then $g(x) = x \log x - x$ so

$$\begin{aligned}
\int (\log x)(\log x) dx &= (\log x)\,[x(\log x) - x] - \int (1/x)[x(\log x) - x] dx \\
&= (\log x)[x(\log x) - x] - \int [\log x - 1] dx \\
&= (\log x)[x(\log x) - x] - \int \log x dx + \int 1 dx \\
&= (\log x)[x(\log x) - x] - [x(\log x) - x] + x \\
&= x(\log x)^2 - 2x(\log x) + 2x.
\end{aligned}$$

[2] Substitution Formula. If f and g' are continuous, then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g'.$$

Example 11.4. (1) Find $\int \frac{1+e^x}{1-e^x} dx$.

Let $u = e^x$; the integral becomes

$$\int \frac{1+u}{1-u} \cdot \frac{1}{u} du = \int \frac{2}{1-u} + \frac{1}{u} du,$$

so

$$\int \frac{1+e^x}{1-e^x} dx = -2\log(1-e^x) + x.$$

(2) Prove that $\int \sqrt{1-x^2} dx = \frac{1}{2} \arcsin x + \frac{1}{2} x\sqrt{1-x^2}$.

(3) A substitution that will transform any rational function involving trigonometric functions into a rational function (see below) is $t = \tan(x/2)$. A simple calculation (that you should carry out) gives $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$. For practice, use this substitution to derive a formula "discovered" by Wright in 1599:

$$\int \sec x dx = \log\left(\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right).$$

Rational Functions

A *rational function* is a quotient of polynomials $\frac{p(x)}{q(x)}$. Notice that over the reals, every odd degree polynomial has a zero (Prove this), so that if

$$q(x) = x^m + b_{m-1}x^{m-1} + \dots + b_0,$$

then

$$q(x) = (x-\alpha_1)^{r_1}\dots(x-\alpha_k)^{r_k}(x^2+\beta_1 x+\gamma_1)^{s_1}\dots(x^2+\beta_\ell x+\gamma_\ell)^{s_\ell},$$

where $m = r_1 + \dots + r_k + 2(s_1 + \dots + s_\ell)$. Notice also that if the degree of p is $\geq m$, then $\frac{p}{q} =$ a polynomial $+\frac{r}{q}$ where the degree of r is strictly less than m.

So we assume that q has the decompoosition given above, and that $p(x) = x^n + \dots$ has degree $n < m$.

Theorem. *The rational function $\frac{p(x)}{q(x)}$ has a decomposition into partial fractions*

$$\begin{aligned}\frac{p(x)}{q(x)} = &\left[\frac{a_{1,1}}{(x-\alpha_1)} + \dots + \frac{a_{1,r_1}}{(x-\alpha_1)^{r_1}}\right] + \dots \\ &+ \left[\frac{a_{k,1}}{(x-\alpha_k)} + \dots + \frac{a_{1,r_k}}{(x-\alpha_k)^{r_k}}\right] \\ &+ \left[\frac{b_{1,1}x + c_{1,1}}{(x^2+\beta_1 x+\gamma_1)} + \dots + \frac{b_{1,s_1}x + c_{1,s_1}}{(x^2+\beta_1 x+\gamma_1)^{s_1}}\right] + \dots \\ &+ \left[\frac{b_{\ell,1}x + c_{\ell,1}}{(x^2+\beta_\ell x+\gamma_\ell)} + \dots + \frac{b_{\ell,s_\ell}x + c_{\ell,s_\ell}}{(x^2+\beta_\ell x+\gamma_\ell)^{s_\ell}}\right].\end{aligned}$$

This impressive–looking theorem is trivial: cross–multiply by q and notice that you can solve the resulting system of linear equations for the a, b, c's on the right hand side.

The main point is of course that each of the expressions on the right hand side can be integrated in elementary terms. Make sure you see how to do this: how, for example, would you compute $\int \frac{1}{x^2+x+1}$? What about $\int \frac{1}{x^2+1}^{27}$?

Example 11.5. (1) Prove that

$$\int \frac{1}{1-x^3}dx = -\frac{1}{3}\log(x-1) + \frac{1}{6}\log(x^2+x+1) + \frac{1}{3}\sqrt{3}\arctan\left(\frac{1}{3}(2x+1)\sqrt{3}\right).$$

(2) Prove that

$$\int \frac{x+4}{x^2+1}dx = \frac{1}{2}\log(x^2+1) + 4\arctan(x).$$

(3) Prove that

$$\begin{aligned}\frac{8}{\sqrt{2}}\int \frac{1}{x^4+1}dx = \log&\left(\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}\right) \\ &+ 2\arctan\left(x\sqrt{2}+1\right) + 2\arctan\left(x\sqrt{2}-1\right).\end{aligned}$$

(4) For those who have encountered the hyperbolic functions:

$$\int \frac{1}{x^4-1} = -\frac{1}{2}\operatorname{arctanh} x - \frac{1}{2}\arctan x.$$

Example 11.6. THE GAMMA FUNCTION. An important classical function is the gamma function $\Gamma(x)$. Prove that

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dx$$

defines a function of x for all $x > 0$ (that is, show that the improper Riemann integral exists). Using integration by parts, show that

$$\Gamma(x+1) = x\Gamma(x), \text{ and } \Gamma(1) = 1.$$

Deduce by induction that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$.

Things you should have learned in this section

(1) Various integration methods.
(2) Statement and proof of the method of integration by parts.
(3) Statement and proof of the method of partial fractions.

TBW/1A22